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A TRANSFORMATION OF THE EQUATIONS OF MECHANICS†

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The structure of the equations obtained from Hamilton's equations by a coordinate Legendre transformation of the Hamilton function is discussed. Examples are considered. © 2003 Elsevier Ltd. All rights reserved.

As is well known, under certain conditions of non-degeneracy a Legendre transformation with respect to the velocities of the Lagrange function enables one to change from Lagrange's equations to Hamilton's equations, while a Lagrange transformation with respect to the momenta of the Hamilton function enables the inverse change to be made, and thereby enables a one-to-one correspondence to be established between the Lagrange and Hamilton descriptions of the dynamics. It turns out that a Legendre transformation of the Hamilton function with respect to the coordinates, which is possible when the non-degeneracy conditions are satisfied, is also interesting. This paper is devoted to a discussion both of the structure of the equations obtained and of the possibility of using them to investigate the dynamics of mechanical systems.

1. THE CONVENTIONAL RELATION BETWEEN THE HAMILTON AND LAGRANGE DESCRIPTIONS OF MOTION

Consider the Hamilton system

$$\dot{\mathbf{q}} = \partial H/\partial \mathbf{p}, \quad \dot{\mathbf{p}} = -\partial H/\partial \mathbf{q}, \quad \mathbf{p}, \mathbf{q} \in R^n; \quad H = H(\mathbf{p}, \mathbf{q}, t) \tag{1.1}$$

In mechanics the Hamilton function H is usually quadratic in the momentum $\mathbf{p}$, and the corresponding quadratic form is usually positive definite. This enables us to carry out a Legendre transformation with respect to the momenta

$$\mathbf{v} = \partial H/\partial \mathbf{p} \tag{1.2}$$

and obtain the inverse mapping

$$\mathbf{p} = \mathbf{p}(\mathbf{v}, \mathbf{q}, t) \tag{1.3}$$

as the solution of Eq. (1.2) for $\mathbf{p}$ and construct the corresponding function

$$L(\mathbf{v}, \mathbf{q}, t) = \mathbf{p} \cdot \mathbf{v} - H(\mathbf{p}, \mathbf{q}, t) \tag{1.4}$$

in which, instead of the quantity $\mathbf{p}$, we have substituted its value (1.3). This function is the Lagrange function for Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}} = \frac{\partial L}{\partial \mathbf{q}} \tag{1.5}$$

They must be supplemented by the first, "kinematic", subsystem of Eqs (1.1), which, in the new notation, has the form

$$\dot{\mathbf{q}} = \mathbf{v} \tag{1.6}$$

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The correctness of the change from the Lagrange to the Hamilton formalism is ensured by the possibility of obtaining, at least locally, one and only one solution of Eqs (1.2), and also the identity

$$\partial L(\mathbf{v}, \mathbf{q}, t)/\partial \mathbf{q} = -\partial H(\mathbf{p}(\mathbf{v}, \mathbf{q}, t), \mathbf{q}, t)/\partial \mathbf{q} \tag{1.7}$$

2. THE LEGENDRE TRANSFORMATION OF THE HAMILTON FUNCTION WITH RESPECT TO THE COORDINATES AND ITS CONSEQUENCES

The variable $\mathbf{p}$ and $\mathbf{q}$ in the Hamilton function look like quantities of the same order of importance. The question arises as to what happens to the Hamilton equations if the Legendre transformation is carried out not with respect to the variables $\mathbf{p}$ but with respect to the variables $\mathbf{q}$. We will carry out this transformation. We have the variables

$$\mathbf{e} = \partial H/\partial \mathbf{q} \tag{2.1}$$

and the conjugate variables $\mathbf{q}$ in the sense of the Legendre transformation. We will assume that the solution of Eqs (2.1), considered as a system in $\mathbf{q}$, has the form

$$\mathbf{q} = \mathbf{q}(\mathbf{p}, \mathbf{e}, t) \tag{2.2}$$

We construct the function

$$\Lambda(\mathbf{p}, \mathbf{e}, t) = \mathbf{e} \cdot \mathbf{q} - H(\mathbf{p}, \mathbf{q}, t) \tag{2.3}$$

in which the quantity $\mathbf{q}$ is replaced by its value (2.2).

Differentiation of the function (2.3) with respect to $\mathbf{p}$ and $\mathbf{e}$ gives

$$\partial \Lambda/\partial \mathbf{p} = -\partial H/\partial \mathbf{p}, \quad \partial \Lambda/\partial \mathbf{e} = \mathbf{q}$$

Then the Hamilton system (1.1) can be rewritten in the form

$$\frac{d}{dt}\frac{\partial \Lambda}{\partial \mathbf{e}} = -\frac{\partial \Lambda}{\partial \mathbf{p}} \tag{2.4}$$

This system must be supplemented by the "kinematic" system

$$\dot{\mathbf{p}} = -\mathbf{e} \tag{2.5}$$

3. AN ALTERNATIVE POSSIBILITY OF TRANSFORMING A HAMILTON SYSTEM

Equations (2.4) and (2.5) differ in their form from the "classical" Lagrange equations (1.5) and (1.6) solely in the signs of the right-hand sides. It turns out that one can represent these equations in a form which is identical with the form of Lagrange's equations. To do this we consider the function

$$H'(\mathbf{p}, \mathbf{q}, t) = -H(\mathbf{p}, \mathbf{q}, t) \tag{3.1}$$

Hamilton's equations can then be written in the form

$$\dot{\mathbf{q}} = -\partial H'/\partial \mathbf{p}, \quad \dot{\mathbf{p}} = \partial H'/\partial \mathbf{q} \tag{3.2}$$

Suppose

$$\mathbf{V} = \partial H'/\partial \mathbf{q} \tag{3.3}$$

and this relation, considered as a system of equation in $\mathbf{q}$, allows of the unique solution

$$\mathbf{q} = \mathbf{q}(\mathbf{p}, \mathbf{V}, t) \tag{3.4}$$

We will construct the function

$$\mathscr{L}(\mathbf{p}, \mathbf{V}, t) = \mathbf{q} \cdot \mathbf{V} - H'(\mathbf{p}, \mathbf{q}, t)$$

Then the following relations hold

$$\partial \mathscr{L}/\partial \mathbf{V} = \mathbf{q}, \quad \partial \mathscr{L}/\partial \mathbf{p} = -\partial H'/\partial \mathbf{p} \tag{3.5}$$

by means of which Eqs (3.2) can be written as

$$\frac{d}{dt}\frac{d\mathscr{L}}{\partial \mathbf{V}} = \frac{\partial \mathscr{L}}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = \mathbf{V} \tag{3.6}$$

or, in the more usual form, as

$$\frac{d}{dt}\frac{d\mathscr{L}}{d\dot{\mathbf{p}}} = \frac{\partial \mathscr{L}}{\partial \mathbf{p}} \tag{3.7}$$

4. THE CONNECTION WITH HAMILTON'S VARIATIONAL PRINCIPLE

It is easy to follow the connection between Eqs (3.7) and Hamilton's variational principle in the Poincaré form. The functional of action can be transformed as follows:

$$\begin{aligned} S &= \int_a^b [\mathbf{p} \cdot \dot{\mathbf{q}} - H(\mathbf{p}, \mathbf{q}, t)]dt = [\mathbf{p} \cdot \mathbf{q}]_a^b - \int_a^b [\mathbf{q} \cdot \dot{\mathbf{p}} + H(\mathbf{p}, \mathbf{q}, t)]dt = \\ &= [\mathbf{p} \cdot \mathbf{q}]_a^b - \int_a^b [\mathbf{q} \cdot \dot{\mathbf{p}} - H'(\mathbf{p}, \mathbf{q}, t)]dt = [\mathbf{p} \cdot \mathbf{q}]_a^b + S' \end{aligned} \tag{4.1}$$

We will vary the functional S' in the class of curves with ends fixed with respect to the variables $\mathbf{p}$ and free with respect to the variables $\mathbf{q}$. By equating the first variation of this functional to zero we obtain exactly Eqs (3.2) and, as a consequence, Eqs (3.7). These discussions are exactly analogous to those when analysing the "classical" Hamilton variational principle in the Poincaré form.

For mechanical systems the Hamilton function is linear-quadratic in the momenta, and its quadratic component is always positive definite and hence non-degenerate. This always enables us to obtain the inverse transformation, which is also, perhaps, the main advantage of the Legendre transformation with respect to the momenta of the Hamilton function. In the case when the Hamilton function is explicitly independent of time, this feature enables one to obtain the Jacobi variational principle in explicit form, which is particularly convenient for a geometrical analysis of the trajectories of mechanical systems in configuration space.

At the same time, a Legendre transformation of the Hamilton function with respect to the coordinates often turns out to be either impossible, in view of the degeneracy, or presents difficulties due to analytical complexities. In such cases we can speak of a transformation with respect to parts of the coordinates, obtaining an analogue of Routh's equations. However, in the case when the Hamilton function is linear-quadratic with respect to the coordinates and the matrix of quadratic form is non-degenerate, it is possible to construct an analogue of Jacobi's principle in momentum space.

5. HAMILTON FUNCTIONS THAT ARE LINEAR-QUADRATIC WITH RESPECT TO THE COORDINATES

An analogue of Jacobi's variational principle. Suppose the Hamilton function is linear-quadratic with respect to the coordinates

$$H = \frac{1}{2}\mathbf{A}\mathbf{q} \cdot \mathbf{q} + \mathbf{B} \cdot \mathbf{q} + \mathbf{C}, \quad \mathbf{A} = \mathbf{A}^T \tag{5.1}$$

where the components of the matrix **A** and of the vector **B**, and also the quantity **C** are continuous functions of the momenta. Suppose

$$\mathbf{V} = \partial H'/\partial \mathbf{q} = -(\mathbf{A}\mathbf{q} + \mathbf{B}), \quad H' = -H \tag{5.2}$$

Then, if the determinant of the matrix **A** is non-zero, we have

$$\mathbf{q} = -\mathbf{A}^{-1}(\mathbf{V} + \mathbf{B}) \tag{5.3}$$

Using this relation we obtain the Lagrange function in the form

$$\mathcal{L}(\mathbf{V}, \mathbf{p}) = (\mathbf{q} \cdot \mathbf{V} - H') = -\mathbf{A}^{-1}(\mathbf{V} + \mathbf{B}) \cdot \mathbf{V} + \frac{1}{2}(\mathbf{V} + \mathbf{B}) \cdot \mathbf{A}^{-1}(\mathbf{V} + \mathbf{B}) - \mathbf{B} \cdot \mathbf{A}^{-1}(\mathbf{V} + \mathbf{B}) + \mathbf{C} =$$

$$= -\frac{1}{2}\mathcal{A}\mathbf{V} \cdot \mathbf{V} - \mathcal{B} \cdot \mathbf{V} + \mathcal{C}, \quad \mathcal{A} = \mathbf{A}^{-1}, \quad \mathcal{B} = \mathbf{A}^{-1}\mathbf{B}, \quad \mathcal{C} = \mathbf{C} - \frac{1}{2}\mathbf{A}^{-1}\mathbf{B} \cdot \mathbf{B}$$

Recalling that $\mathbf{V} = \dot{\mathbf{p}}$, we can represent the energy integral (the Painlevé–Jacobi integral) in the form

$$\mathcal{T} = \frac{\partial \mathcal{L}}{\partial \mathbf{V}} \cdot \mathbf{V} - \mathcal{L} = -\frac{1}{2}\mathcal{A}\dot{\mathbf{p}} \cdot \dot{\mathbf{p}} - \mathcal{C} = -h \tag{5.4}$$

which exists by virtue of the fact that the function $\mathcal{L}$ is explicitly independent of time. We have from relation (5.4)

$$dt^2 = \frac{\mathbf{A}d\mathbf{p} \cdot d\mathbf{p}}{2(h - \mathcal{C})} \tag{5.5}$$

Then, in the class of curves in momentum space, emerging from the point $\mathbf{p}_a$, which arrive at the point $\mathbf{p}_b$ and which satisfy relation (5.4), the functional S', by virtue of (5.5), allows of the following representation (compare with [1–3])

$$\begin{aligned} S' &= -\int_a^b \mathcal{L}dt = -h(b - a) + S'' \\ S'' &= -\int_a^b \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{p}}} \cdot \dot{\mathbf{p}}dt = \int_a^b (\mathcal{A}\dot{\mathbf{p}} + \mathcal{B}) \cdot \dot{\mathbf{p}}dt = \int_{p_a}^{p_b} (2(h - \mathcal{C})\mathcal{A}d\mathbf{p} \cdot d\mathbf{p})^{1/2} + \mathcal{B} \cdot d\mathbf{p} \end{aligned} \tag{5.6}$$

If the initial quadratic form, specified by the matrix **A**, is positive definite, then the quadratic form specified by the matrix $\mathcal{A}$ is also positive definite, and the "Jacobi metric in momentum space", defined by relation (5.6), has the same properties as the usual Jacobi metric in configuration space (see, for example, [4, Chapter 6] and also [5, Chapter 3]). However, in general, such positive definiteness does not occur.

For the positive definite matrix $\mathcal{A}$ the region of possible motion is defined in the usual way: this is the set of points $\mathbf{p} \in \Pi_h$ such that

$$\Pi_h = \left\{\mathbf{p}: 0 \le h - \mathcal{C}\left(= \frac{1}{2}\mathcal{A}\dot{\mathbf{p}} \cdot \dot{\mathbf{p}}\right)\right\}$$

For a negative definite matrix $\mathcal{A}$ the area of the possible motion is defined as

$$\Pi_h = \left\{\mathbf{p}: \left(\frac{1}{2}\mathcal{A}\dot{\mathbf{p}} \cdot \dot{\mathbf{p}} =\right)h - \mathcal{C} \le 0\right\}$$

When the quadratic form defined by the matrix $\mathcal{A}$ is sign indefinite, it is difficult to define the region of possible motion correctly. Nevertheless, if

$$\mathcal{A}\dot{\mathbf{p}} \cdot \dot{\mathbf{p}} = A^+ - A^- = a_1p_1^2 + a_2p_2^2 + \ldots + a_kp_k^2 - a_{k+1}p_{k+1}^2 - a_{k+2}p_{k+2}^2 - \ldots - a_np_n^2$$

$$a_i > 0, \; i = 1, 2, \ldots$$

the generalised region of possible motion is given by the inequalities

$$-A^{-} \leq h - \mathscr{C} \leq A^{+}$$

An investigation of the properties of the generalized region of possible motion defined in this way is outside the scope of the present paper.

6. EXAMPLES

The motion of a particle in a field with a quadratic potential. The dynamics of a particle in a field with a quadratic potential can be described by the classical Lagrange equations with Lagrangian

$$L = \frac{1}{2}(\mathbf{v}^2 - \mathscr{A}\mathbf{q} \cdot \mathbf{q}); \quad \mathscr{A} = \operatorname{diag}(\Omega_1, \Omega_2, \ldots, \Omega_n), \quad \Omega_i = \pm\omega_i^2$$

These equations have the form

$$\ddot{\mathbf{q}} = -\mathscr{A}\mathbf{q} \Leftrightarrow \begin{cases} \dot{\mathbf{v}} = -\mathscr{A}\mathbf{q} \\ \dot{\mathbf{q}} = \mathbf{v} \end{cases}$$

Hamilton's equations with the Hamilton function

$$H = \frac{1}{2}(\mathbf{p}^2 + \mathscr{A}\mathbf{q} \cdot \mathbf{q}) \tag{6.1}$$

have the form

$$\dot{\mathbf{q}} = \mathbf{p}, \quad \dot{\mathbf{p}} = -\mathscr{A}\mathbf{q}$$

By virtue of a Legendre transformation of the function (6.1) with respect to the variable $\mathbf{q}$

$$\mathbf{e} = \partial H / \partial \mathbf{q} = \mathscr{A}\mathbf{q} \Leftrightarrow \mathbf{q} = \mathscr{A}^{-1}\mathbf{e} \tag{6.2}$$

the function Λ can be written in the form

$$\Lambda = \frac{1}{2}(\mathscr{A}^{-1}\mathbf{e} \cdot \mathbf{e} - \mathbf{p}^2) \tag{6.3}$$

Here Eqs (2.4) and (2.5) can be represented as

$$\mathscr{A}^{-1}\dot{\mathbf{e}} = \mathbf{p}, \quad \dot{\mathbf{p}} = -\mathbf{e}$$

or

$$-\mathscr{A}^{-1}\ddot{\mathbf{p}} = \mathbf{p} \tag{6.4}$$

The last equation can also be obtained by a Legendre transformation of the function $H' = -H$, which has the form

$$\mathbf{V} = \partial H' / \partial \mathbf{q} = -\mathscr{A}\mathbf{q}, \quad \Leftrightarrow \mathbf{q} = -\mathscr{A}^{-1}\mathbf{V} \ (\mathbf{V} = \dot{\mathbf{p}}) \tag{6.5}$$

by virtue of which

$$\mathscr{L} = [\mathbf{V} \cdot \mathbf{q} - H']_{(6.5)} = -\dot{\mathbf{p}} \cdot \mathscr{A}^{-1}\dot{\mathbf{p}} + \frac{1}{2}[\mathbf{p}^2 + \mathscr{A}^{-1}\dot{\mathbf{p}} \cdot \dot{\mathbf{p}}] = \frac{1}{2}[\mathbf{p}^2 - \mathscr{A}^{-1}\dot{\mathbf{p}} \cdot \mathbf{p}] \tag{6.6}$$

The Lagrange equations with the Lagrange function (6.6) are identical with Eqs (6.4).

A relativistic particle in a field with a quadratic potential. The dynamics of a relativistic particle in a three-dimensional Euclidean space under the action of a linear potential force can be described by Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}} = \frac{\partial L}{\partial \mathbf{q}}, \quad \dot{\mathbf{q}} = \mathbf{v}; \quad \mathbf{q}, \mathbf{v} \in R^3$$

with the Lagrange function (see, for example, [2, Chapter 3])

$$L = -mc^2\left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{1/2} - \mathscr{C}, \quad \mathscr{C} = \frac{1}{2}(\mathbf{Aq}, \mathbf{q}), \quad \mathbf{A} = \operatorname{diag}(A_1, A_2, A_3) \tag{6.7}$$

These equations have the form

$$\frac{d}{dt}\frac{m\mathbf{v}}{(1 - \mathbf{v}^2 c^{-2})^{1/2}} = -\mathbf{Aq}, \quad \dot{\mathbf{q}} = \mathbf{v}$$

The dynamics of this system can be described by Hamilton's equations

$$\dot{\mathbf{q}} = \partial H/\partial \mathbf{p}, \quad \dot{\mathbf{p}} = -\partial H/\partial \mathbf{q}, \quad H = c(m^2c^2 + \mathbf{p}^2)^{1/2} + \mathscr{C}$$

These equations have the form

$$\dot{\mathbf{q}} = \frac{c\mathbf{p}}{(m^2c^2 + \mathbf{p}^2)^{1/2}}, \quad \dot{\mathbf{p}} = -\mathbf{Aq}$$

We carry out a Lagendre transformation of the function $H' = -H$ with respect to the variable $\mathbf{q}$. We have

$$\mathbf{V} = \partial H'/\partial \mathbf{q} = -\mathbf{Aq}$$

i.e.

$$\mathbf{q} = -\mathbf{A}^{-1}\mathbf{V}$$

Then, bearing in mind that $\mathbf{V} = \dot{\mathbf{p}}$, we can represent the Lagrange function in the form

$$\mathscr{L} = \mathbf{V}\cdot\mathbf{q} - H' = -\frac{1}{2}\mathbf{A}^{-1}\dot{\mathbf{p}}\cdot\dot{\mathbf{p}} + c(m^2c^2 + \mathbf{p}^2)^{1/2}$$

Here Eqs (3.7) can be written as

$$-\mathbf{A}^{-1}\ddot{\mathbf{p}} = c\frac{\mathbf{p}}{(m^2c^2 + \mathbf{p}^2)^{1/2}}$$

Unlike the classical multidimensional oscillator, the integrability, as well as the non-integrability, of a relativistic top is not obvious, if all the values of A_i are different.

The classical Jacobi metric in configuration space for this problem is determined as follows. Since the Lagrange function (6.7) is explicitly independent of time, we have the energy integral (the Painlevé–Jacobi integral)

$$\mathscr{T}_{\mathbf{q}} = \frac{\partial L}{\partial \dot{\mathbf{q}}}\cdot\dot{\mathbf{q}} - L = \frac{mc^2}{(1 - \mathbf{v}^2/c^2)} + \mathscr{C} = h$$

by virtue of which

$$dt^2 = \left(c^2\left[1 - \left(\frac{mc^2}{h - \mathscr{C}}\right)^2\right]\right)^{-1} d\mathbf{q}^2$$

Then the shortened action can be represented in the form

$$S'_{\mathbf{q}} = \int_a^b \frac{\partial L}{\partial \dot{\mathbf{q}}}\cdot\dot{\mathbf{q}}\,dt = \int_a^b \frac{m\dot{\mathbf{q}}^2}{(1 - \dot{\mathbf{q}}^2/c^2)^{1/2}}dt = \int_{\mathbf{q}_a}^{\mathbf{q}_b} ds_{\mathbf{q}}$$

where $ds_{\mathbf{q}}^2$ is the Jacobi matrix in configuration space, given by

$$ds_{\mathbf{q}}^2 = \frac{(h-\mathscr{E})^2-(mc^2)^2}{c^2}d\mathbf{q}^2 \tag{6.8}$$

At the same time, the Jacobi metric in momentum space has the form

$$ds_{\mathbf{q}}^2 = 2(h-c(m^2c^2+\mathbf{p}^2)\mathbf{A}^{-1}d\mathbf{p}\cdot d\mathbf{p})$$

A particle in Hamel mechanics. Hamel [7, pp. 316, 317], when discussing variational methods of describing relativistic mechanics, proposed to consider, in addition to mechanical systems described in classical mechanics by Lagrange's equations with Lagrangian $L = L(\mathbf{q}, \dot{\mathbf{q}})$, Lagrangian systems with Lagrangian

$$\mathscr{H} = \sqrt{E-2L} \tag{6.9}$$

where $E > 0$ is a constant having the dimensions of energy. If $E \gg 2L$, Hamel's function $\mathscr{H}$ can be expanded in series in powers of the parameter $2L/E$

$$\mathscr{H} = \mathscr{H}_0+\mathscr{H}_1+\mathscr{H}_2+\ldots = \sqrt{E}\left(1-\frac{L}{E}-\frac{L^2}{2E^2}+\ldots\right)$$

The first term of the expansion is a constant. The second term, apart from the factor, which plays no role for the equations of motion, is identical with the Lagrangian of the initial problem of classical mechanics. The difference from the equations of classical mechanics, and, under certain conditions, from the post-classical approximation of relativistic mechanics also, begins to manifest itself due to the terms $\mathscr{H}_2$. Some general properties of Hamel mechanics were investigated in [8].

Within the framework of Hamel mechanics we will consider the motion of a particle in a field with a quadratic potential. The equations of motion have the form

$$\frac{d}{dt}\frac{1}{\sqrt{E-2L}}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{1}{\sqrt{E-2L}}\frac{\partial L}{\partial \mathbf{q}} \tag{6.10}$$

where, if

$$L = \frac{1}{2}(\mathbf{v}^2-\mathbf{A}\mathbf{q}\cdot\mathbf{q})$$

They can be represented in the form of a system of Hamilton equations with Hamilton function [8]

$$H = -G(1+\mathbf{p}^2)^{1/2}, \quad G = (E+\mathbf{A}\mathbf{q}\cdot\mathbf{q})^{1/2}$$

The Lagrange transformation of the function $H' = -H$ with respect to the coordinates has the form

$$\mathbf{V} = \mathbf{A}\mathbf{q}(1+\mathbf{p}^2)^{1/2}/G \tag{6.11}$$

Then

$$\mathbf{A}^{-1}\mathbf{V} = \mathbf{q}(1+\mathbf{p}^2)^{1/2}/G \tag{6.12}$$

Multiplying the left- and right-hand sides of Eqs (6.11) and (6.12) scalarly, we arrive at a relation from which it follows that

$$\mathbf{A}\mathbf{q}\cdot\mathbf{q} = \frac{E\mathbf{A}^{-1}\mathbf{V}\cdot\mathbf{V}}{1+\mathbf{p}^2-\mathbf{A}^{-1}\mathbf{V}\cdot\mathbf{V}}, \quad E+\mathbf{A}\mathbf{q}\cdot\mathbf{q} = \frac{E(1+\mathbf{p}^2)}{1+\mathbf{p}^2-\mathbf{A}^{-1}\mathbf{V}\cdot\mathbf{V}}$$

This enables us to write the Lagrange function in the form

$$\mathcal{L} = -(1+\mathbf{p}^2)^2 E/G = -E^{1/2}(1+\mathbf{p}^2-\mathbf{A}^{-1}\mathbf{p}\cdot\mathbf{p})^{1/2}$$

For this problem the Jacobi metric in configuration space and the Jacobi metric in momentum space can be obtained in the same ways as was done when finding the Jacobi metric in configuration space for a relativistic particle.

The motion of a particle in a central force field. Consider the motion of a system described by Hamilton's equations with the Hamilton function

$$H = \frac{1}{2}\mathbf{p}^2 + c(\mathbf{x}\cdot\mathbf{x})^{\alpha/2}$$

If the constant c is equal to the product of the Newtonian constant and the mass of the attracting centre and $\alpha = 1$, we have the classical Kepler problem. The Legendre transformation with respect to the coordinates of the function $H' = -H$ has the form

$$\mathbf{V} = \partial H'/\partial\mathbf{x} = -c\alpha\mathbf{x}(\mathbf{x}\cdot\mathbf{x})^{\alpha/2-1}$$

whence

$$\mathbf{V}\cdot\mathbf{V} = c^2\alpha^2(\mathbf{x}\cdot\mathbf{x})^{\alpha-1}, \quad \mathbf{x}\cdot\mathbf{x} = [c^{-2}\alpha^{-2}\mathbf{V}\cdot\mathbf{V}]^{1/(\alpha-1)}$$

These relations enable us to obtain the Lagrange function

$$\mathcal{L} = c^{1/(\alpha-1)}(1-\alpha)\alpha^{\alpha/(\alpha-1)}(\dot{\mathbf{p}}\cdot\dot{\mathbf{p}})^{\alpha/(2(\alpha-1))} + \frac{1}{2}\mathbf{p}^2$$

However, for this problem the advantages of describing the motion within the framework of the formalism described above are not so obvious.

Remark. Variables of the type

$$\mathbf{V} = -\partial H/\partial\mathbf{q} = \partial H'/\partial\mathbf{q}$$

which arise in the problem for systems with an elastic potential are similar to stresses. The description of the dynamics of systems "in stresses" is well known in continuum mechanics.

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